SYNTHESIS OF COMPLEX NETWORKS
REGULAR FRACTALS

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Abstract. The article is sacred to the questions of design of topology of complex networks with the use of new methods of fractal geometry. A complex network structure is in-process synthesized, through replacement of tops of count by a count-fuse, and also the simplest, including affine transformations. The distinguishing feature of pre-fractal counts is possibility of receipt of numeral descriptions of network, coming from the analysis of properties of his counts-fuses. For the got pre-fractal count properties of fractal and scale invariance are investigational. An example of calculation of fractal dimension of pre-fractal count of complex network is made at the use of square and three-cornered blocks for coverage of network. It is shown that for the effective analysis of properties self-similarity of pre-fractal counts it is necessary to use V-coverage. Dependence of fractal dimension of pre-fractal count is investigational on the linear size of grate \( \varepsilon \)-coverage with the purpose of determining the amount of factors, influencing on the system. They got results are actual at the decision of intricate problems.

Keywords: scale-invariant complex network, network topology, pre-fractal graphs, fractal dimension, self-similarity.

Introduction

Current trends of development of scale-invariant complex networks indicate about the need to develop effective methods of modeling and optimization of networks of large sizes. Possibilities of applying to complex networks, which have inherent continuous expansion and dynamic characteristics, of the ordinary methods of modeling are significantly limited. In complex telecommunications networks with a plurality of subscriber stations appear completely new properties, such as survivability, reliability, alternativeness of delivery routes of messages to the recipient, instability, conflict, etc. Properties of this class of networks depend strongly on their geometry (topology) (1). Thus, one of the fundamental characteristics of a complex network is their topological dimension. Have calculated the dimension of the network topology we can define in quantity of system network properties and find general information laws of motion data streams.

Taking to attention that telecommunications networks are composed of many interconnected networks of lower rank, the elements of which are extremely varied and complex, for their research use modeling fractals generated by special algorithms. Such artificial fractal objects are called "regular fractals" (2).

Studies on fractal synthesis and analysis of the topology of complex information networks (e.g., modeling fractal and pre-fractal graphs, calculation of their fractal dimension) are described in both domestic and foreign sources (4-10). Among these research papers worth of noting are those (6, 7), in which the characteristics of pre-fractal connected graphs are evaluated.

The properties of self-similarity allow forecasting the ranges for quantitative estimates for the test characteristics of the network.

In study (8) one of the possible rules is considered that specifies the structural dynamics of complex multi-element network systems. Formal representations of network systems structures change according to this rule are scale-invariant or self-similar structures.

In studies (9, 10) the problems of fractal modeling and multiobjective optimization are discussed arising in the design and analysis of computer network system. The approach suggested is based on the method of constructing of the set of non-dominated alternatives and on the method of constructing of logical functions of the variable significance of data containing the information of alternatives, which serve as typical topological structures of network.
In most studies on this issue, the authors don’t emphasize that the strict definition of the fractal dimension is useful for analyzing the topology of scale-invariant networks and tend to decide one of two problems: either the synthesis of fractal network structures or calculation of the fractal dimension of the system or process.

That’s why the aim of this study is to design of the structure of the scale-invariant network possessing of properties of self-similarity.

**Method**

**I stage. Synthesis of complex network topology (direct problem).**

The network graph is given \( G = (V, E) \), where \( V \) - set of vertices, \( E \) is the set of edges and the graph-seed \( H = (W, Q) \), where \( W \) - set of vertices of graph-seed, \( Q \) - set of its edges.

Have to recursively define pre-fractal graph \( G_L = (V_L, E_L) \), step by step changing in graph \( G_l = (V_l, E_l) \) each its top by seed \( H = (W, Q) \), where \( l = 1, 2, ..., L - 1 \), where \( l \) is the amount of steps, \( L \) - the amount of ranks.

**Restriction.** Vertices are joined arbitrarily (randomly) or by a certain rule, using affine transformations that form and space groups and in which, unlike projective transformations, the principle of duality (rotate, shift, reflection, compression and tensile of seed as geometric object) don’t act.

Generation process of pre-fractal graph \( G_L \) is the process of constructing the sequence of pre-fractal graphs \( G_1, G_2, ..., G_L \) - called trajectory. Fractal graph \( G = (V, E) \), generated by seed \( H = (W, Q) \), is determined by the infinite of trajectory.

In given graph in the top planned for replace \( V \in V \) will distinguish the set \( \{V_j\} \leq V \) with \( j = 1, 2, ..., |\tilde{V}| \) of vertices adjacent to it. Next, remove from \( G \) the vertex \( \tilde{v} \) and all incident edges. Then each vertex \( \tilde{v}_j \in \tilde{V} \), \( j = 1, 2, ..., |\tilde{V}| \), will join by an edge with one of the vertex’s of seed \( H = (W, Q) \).

On stage \( l = 1 \) the seed corresponds to the pre-fractal graph \( G_1 = H \). About process described say that pre-fractal graph \( G_L = (V_L, E_L) \) is generated by seed \( H = (W, Q) \).

Using the manipulations of top replacement by seed (4) in the process of generating of pre-fractal graph \( G_L \), for the elements \( G_l = (V_l, E_l) \), \( l \in \{1, 2, ..., L - 1\} \) of its trajectory, allows to introduce the mapping \( \varphi : V_l \rightarrow V_{l+1} \) or \( \varphi(V_l) = V_{l+1} \), in general:

\[
\varphi^t(V_l) = V_{l+t}, \ t = 1, 2, ..., L - l, \tag{1}
\]

where \( V_{l+t} \) - the image of \( V_l \), \( t \) - stage number of generation of pre-fractal graph.

For pre-fractal graph \( G_L \), ribs, appeared on the \( l \)-th, \( l \in \{1, 2, ..., L\} \) stage of generation will be called edges of rank \( l \). New ribs of pre-fractal graph \( G_L \) will be called ribs of rank \( L \), and all other edges are called-old ones (4).

If to remove from the pre-fractal graph \( G_L \), generated by \( n \)-vertexes seed \( H \), consistently all old edges (edges of rank \( l \), \( l = 1, 2, ..., L - 1 \)), then the original graph splits into many disconnected components \( \{B_L^{(i)}\} \), each of which is isomorphic to the seed \( H \). Many components \( \{B_L^{(i)}\} \) will be called blocks of the first rank. Similarly, when you remove from pre-fractal graph \( G_L \) all old ribs of ranks \( l = 1, 2, ..., L - 2 \), we will obtain a plurality of blocks \( \{B_L^{(2)}\} \) of the second rank. Summarizing, we note that when you delete from pre-fractal \( G_L \) all edges of all ranks \( l = 1, 2, ..., L - r \), we obtain a set \( \{B_L^{(r)}\} \), \( r \in \{1, 2, ..., L - 1\} \), block \( r \)-th rank, where \( i = 1, 2, ..., n^{L-r} \) - number of unit. Blocks
$B^{(1)}_L \subseteq G_L$ with the first rank will also be called sub-graphs-seeds $H$ of pre-fractal graph $G_L$.

Obviously, that every block of $B^{(r)}_L = \left( U^{(r)}_L, M^{(r)}_L \right)$, $r \in \{1,2,\ldots,L-1\}$ is pre-fractal graph $B_r = (U_r, M_r)$, generated by seed $H$ (5).

Specify mapping $\varphi$ in (1): for any vertex $v_j \in V_l$, $j \in \{1,2,\ldots,n^l\}$, of pre-fractal graph $G_l = (V_l, E_l)$, $l \in \{1,2,\ldots,L-1\}$ from the path of graph $G_L$, rightly

$$\varphi^l(v_j) = U^{(t)}_{l+t,j}, \quad \varphi^l(v_j) = B^{(t)}_{l+t,j}, \quad (2)$$

where $B^{(t)}_{l+t,j} = \left( U^{(t)}_{l+t,j}, M^{(t)}_{l+t,j} \right) \subseteq G_{l+t}, \quad t = 1,2,\ldots,L-l$.

Similarly,

$$\varphi^l(U^{(r)}_{i,i}) = U^{(r+t)}_{l+t,i}, \quad \varphi^l(B^{(r)}_{i,i}) = B^{(r+t)}_{l+t,i}, \quad (3)$$

where $r \in \{1,2,\ldots,L-l\}, \quad i \in \{1,2,\ldots,n^{l-r}\}$.

Example of formation of pre-fractal graph is shown in Fig. 1.

Using seed $G_1 = H$ (Fig. 1.a), as well as replacement operation of vertex by the sub-graph-seed on the second stage, we obtain a set of blocks $\{B^{(2)}_L\}$ of the second rank. Block $B^{(1)}_L$ is the seed of the first rank of pre-fractal graph $G_L$. Accordingly, in the third step we obtain a plurality of blocks $\{B^{(3)}_L\}$ of the third rank (Fig. 1. c).

On (Fig. 1) by bold lines seed edges are marked of 1st rank. The lines of average thickness - ribs of seed of second rank, and thin lines, respectively depict the ribs of seed of third rank. Connected graph $G_1 = H$ is a seed of pre-fractal network graph.

Generalization of described process of formation of pre-fractal graph $G_L = (V_L, E_L)$ is a case when instead of a single seed $H = (W, Q)$ many seeds are used.

Using the operation of seed top replacement, which sense is described above, by the principle of self-similarity, we obtain pre-fractal graph of corresponding rank.

It is worth to note that while the generation of seed of third rank from pre-fractal graph sequentially old ribs removes. To produce the final pre-fractal graph need to spend 6 operations of vertices replacement by seeds (in bold line) (Fig. 2).
Stage II. Checking the network topology on fractality.

Pre-fractal graph is given \( G_L = (V_L, E_L) \), represented by any non empty set of \( n \)-dimensional Euclidean space \( G^n \).

**Restriction.** The diameter of \( G \) is defined as \( \text{diam}(G) = \text{max} \{ \| x - y \| : x, y \in G \} \) (the largest the distance between any pair of segments \( G \)). If \( \{ G_i \} \) – countable (or finite) set of open sets of diameter not greater than \( \varepsilon \), which covers \( P \), \( P \subset \bigcup_{i=1}^{\infty} G_i \), with \( 0 < |G_i| \leq \varepsilon \) for each \( i \), then \( \{ G_i \} \) is an \( \varepsilon \)-covering of \( P \).

**Necessary** for any \( \varepsilon > 0 \) to define \( H^S_0(P) = \inf \left\{ \sum_{i=1}^{\infty} |G_i|^S \right\} \), where \( \{ G_i \} \) – covering of \( P \); \( S \) – non negative number; let \( C(\varepsilon) \) – to be the minimum number of triangles (or squares) with \( \varepsilon \), area required to cover \( P \) and calculate the fractal dimension of amount \( D_G \).

**Restriction.** With decreasing of \( \varepsilon \) meaning the class of admissible coverings of \( P \) decreases. The value of \( \inf H^S_0(P) \) increases and reaches the limit, when \( \varepsilon \to 0 \).

In the analysis of the fractal properties of the network is necessary to coordinate topology of simulated network with the geometry of the object that it describes. Telecommunication networks, according to possible large connectivity suggest plurality of delivery routes messages to the destination. This affects the fractal dimension.

To check the properties of self-similarity of the network, let’s define the capacitive fractal dimension. For the obtained pre-fractal graph \( G = (V, E) \) (Fig. 2), which describes the topology of the network, we can determine the dimension of \( D_G \) by follow means:

\[
D_G = \lim_{\varepsilon \to 0} \frac{\ln(C_G(\varepsilon))}{\ln(\varepsilon)}
\]

where \( C_G(\varepsilon) \) - the number of blocks from the long side, equal \( \varepsilon \), needed to cover \( G = (V, E) \), as set of seeds, each block must cover a single seed (Fig. 3) [6].

On some certain stage of covering of pre-fractal graph \( C(\varepsilon) \) of squares with side \( \varepsilon \), and on next – \( C(\varepsilon^*) \) elements with side \( \varepsilon^* \) were used. Taking to attention suggested power-law dependence it is true:

\[
C(\varepsilon) \cong \frac{1}{\varepsilon^{D_G}}, \quad C(\varepsilon^*) \cong \frac{1}{\varepsilon^{*D_G}}.
\]
Fig. 3 Pre-fractal graph is covered by square blocks with cells dimension:

a) $\varepsilon = 30$; b) $\varepsilon = 10$; c) $\varepsilon = 4$

Where the value of $D_G(\varepsilon/\varepsilon^*)$ can be estimated as:

$$ D_G = -\frac{\ln(C(\varepsilon)/C(\varepsilon^*))}{\ln(\varepsilon/\varepsilon^*)}. $$ (5)

In accordance with (5) the value of the fractal dimension $D_G$ (Table 1) is calculated.

Effective application of the covering in this case is to replace the square blocks on the geometrical construction generated by Delaunay triangulation [12], which is optimal for image processing and analysis. Seed of pre-fractal graph $G = (V, E)$ at this is referred as the set of $n \geq 3$ points that do not lie on one line.

In accordance with this, pre-fractal graph was covered by the triangulation covering with cells size $\varepsilon = 30, 10, 4$ (Fig. 4), and the value of the fractal dimension $D_G$ is calculated.

For considered example values $C(\varepsilon)$ depending on the cell size of covering are shown in Table. 1.

Fig. 4 Trig method of covering of topology of network with cells dimensions:

a) $\varepsilon = 30$; b) $\varepsilon = 10$; c) $\varepsilon = 4$

Results

When covering of the test pre-fractal graph by geometric structures which initially is similar to the seed by its form and sizes it is possible much more precisely to define the fractal dimension of synthesized pre-fractal graph.

The data obtained allows monitoring the dynamics of change of fractal dimension obtained of pre-fractal graph depending on the rate of change of $\varepsilon$, and the elements of the cover can be interpreted as the area of clusters, in which transit nodes include at the time of data transfer.
Table 1

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Square Covering</th>
<th>Triangulation covering</th>
<th>V-form covering (size and form of seed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^*$</td>
<td>8</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$\varepsilon/\varepsilon^*$</td>
<td>1</td>
<td>1.6</td>
<td>2</td>
</tr>
<tr>
<td>$C(\varepsilon)$</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$C(\varepsilon)/C(\varepsilon^*)$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.5</td>
</tr>
<tr>
<td>$D_G(\varepsilon)$</td>
<td>2,94</td>
<td>2,58</td>
<td>2,34</td>
</tr>
</tbody>
</table>

Discussion

The study carried out modeling of the complex network using regular fractal objects. For analysis of synthetic topology methodology of calculating the fractal $s$-dimension was used, where $E$ - covering of test object. Description of complex networks by regular fractals allows detecting the properties of self-similarity and scaling invariance of statistical network features. Modeling of pre-fractal graphs by set of seeds reduces the time to design complex networks and allows approximating their models to real networks.

As elements of the pre-fractal graph it is necessary to consider its sub-graph-seeds or, depending on the task, blocks of various ranks. Analysis of the state of the entire structure only for its individual elements significantly reduces the complexity of solving problems associated with complex structures and systems.

For a more accurate estimate of the fractal dimension of the network topology it is advisable to use the method of covering of the tested object based on the geometrical structure of such a size and shape as a seed, which allows to determine the range of variation of the area ratio of fractal coating that preserve self-similarity network.

References


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